Rigidity of holomorphic generators and one-parameter semigroups

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Abstract

In this paper we establish a rigidity property of holomorphic generators by using their local behavior at a boundary point τ of the open unit disk Δ. Namely, if $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ is the generator of a one-parameter continuous semigroup $\{F_t\}_{t\geq 0}$, we state that the equality $f(z) = o\left(|z-\tau|^3\right)$ when $z \to \tau$ in each non-tangential approach region at τ implies that f vanishes identically on Δ. Note, that if F is a self-mapping of Δ then $f = I - F$ is a generator, so our result extends the boundary version of the Schwarz Lemma obtained by D. Burns and

S. Krantz. We also prove that two semigroups $\{F_t\}_{t\geq 0}$ and $\{G_t\}_{t\geq 0}$, with generators f and g respectively, commute if and only if the equality $f = \alpha g$ holds for some complex constant α. This fact gives simple conditions on the generators of two commuting semigroups at their common null point τ under which the semigroups coincide identically on Δ.

1 Introduction.

Let $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H = \{z \in \mathbb{C} : \ \operatorname{Re} z > 0\}$ be the right half-plane. We denote by $\mathrm{Hol}(\Delta, D)$ the set of all holomorphic functions on Δ which map Δ into a set $D \subset \mathbb{C}$, and by $\mathrm{Hol}(\Delta)$ the set of all holomorphic self-mappings of Δ, i.e., $\mathrm{Hol}(\Delta) = \mathrm{Hol}(\Delta, \Delta)$.

The problem of finding conditions for a holomorphic function F to coincide identically with a given holomorphic function G when they have a similar behavior on some subset of $\overline{\Delta}$, has been studied by many mathematicians.

The following assertions are classical:

• *If F and G are holomorphic in Δ and $F = G$ on a subset of Δ that has a nonisolated point, then $F \equiv G$ on Δ (Vitali's uniqueness principle).*

• *If F and G are holomorphic in Δ and continuous on $\overline{\Delta}$, and $F = G$ on some arc γ of the boundary $\partial\Delta$, then $F \equiv G$ on Δ.*

In the point of view of complex dynamics it is natural to study conditions on derivatives of F and G at specific points to conclude that $F \equiv G$.

If, for example, G is the identity mapping I and $\tau \in \Delta$ is the Denjoy–Wolff point of $F \in \mathrm{Hol}(\Delta)$, then the equalities $F(\tau) = G(\tau)$ and $F'(\tau) = G'(\tau)$ provide $F \equiv G$ by the Schwarz Lemma. The same conclusion holds for an arbitrary holomorphic function G on Δ, if F commutes with G and satisfies the conditions $F(\tau) = G(\tau) = \tau$ and $F'(\tau) = G'(\tau) \neq 0$ (see, for instance, [9], [6]).

Different "identity principles" have been recently studied by several mathematicians under suitable boundary conditions. In general, the following three cases are considered.

(A) G is the identity mapping;

(B) G is an arbitrary self-mapping of Δ, and F commutes with G, i.e., $F \circ G = G \circ F$;

(C) G is a constant mapping.

Regarding Case A the following result is due to D. Burns and S. Krantz.

Theorem A ([7]) *Let* $F \in \mathrm{Hol}(\Delta)$ *and*

$$F(z) = 1 + (z-1) + O\left((z-1)^4\right). \tag{1}$$

Then $F \equiv I$*.*

For Case B a uniqueness theorem was given by R. Tauraso in [17] (see also [6]). To formulate this result we need the following notation. Let $F \in \mathrm{Hol}(\Delta)$ and $\tau \in \partial\Delta$. We say that $F \in C_K^m(\tau)$ if it admits the following representation

$$F(z) = \tau + F'(\tau)(z-\tau) + \ldots + \frac{F^{(m)}(\tau)}{m!}(z-\tau)^m + o\left(|z-\tau|^m\right)$$

when $z \to \tau$ in each non-tangential approach region at τ. Moreover, we say that $F \in C^m(\tau)$ if the limit is taken in the full disk.

Theorem B ([17]) *Let* $F, G \in \mathrm{Hol}(\Delta)$ *be commuting functions with a common Denjoy–Wolff point* $\tau \in \partial\Delta$*. If one of the following conditions holds then* $F \equiv G$*.*

(i) $F'(\tau) = G'(\tau) < 1$*;*

(ii) $F \in C^2(\tau)$, $G \in C_K^2(\tau)$, $F'(\tau) = 1$, $F''(\tau) = G''(\tau) \neq 0$ *and* $\mathrm{Re}\,\tau F''(\tau) > 0$*;*

(iii) $F, G \in C^2(\tau)$, $F'(\tau) = 1$, $F''(\tau) = G''(\tau) \neq 0$ *and* $\mathrm{Re}\,\tau F''(\tau) = 0$*;*

(iv) $F \in C^3(\tau)$, $G \in C_K^3(\tau)$, $F'(\tau) = 1$, $F''(\tau) = G''(\tau) = 0$ *and* $F'''(\tau) = G'''(\tau)$*.*

For Case C, when G is a constant mapping, the following fact is an immediate consequence of the Julia–Wolff–Carathéodory Theorem.

- *If* $F \in \mathrm{Hol}(\Delta, \overline{\Delta})$*, then the conditions* $\lim\limits_{r\to1^-} F(r\tau) = \tau$ *and* $\lim\limits_{r\to1^-} F'(r\tau) = 0$ *for some* $\tau \in \partial\Delta$ *imply that* $F \equiv \tau$*.*

In fact, the considering of holomorphic functions f which are not necessarily self-mappings is more relevant in this situation. Various results in this direction were established by S. Migliorini and F. Vlacci in [13].

In what follows we denote by symbol $\angle\lim\limits_{z\to\tau}$ the angular limit of a function defined in Δ at a boundary point $\tau \in \partial\Delta$.

Theorem C (see [13]) *Let $\tau \in \partial\Delta$.*

If $f \in \mathrm{Hol}(\Delta, \overline{H})$, then

$$\angle \lim_{z\to\tau} \frac{f(z)}{z-\tau} = 0 \tag{2}$$

implies that $f \equiv 0$.

More general, if $f \in \mathrm{Hol}(\Delta, \mathbb{C})$, and $f(\Delta)$ is contained in a wedge of angle $\pi\alpha$, $0 < \alpha \leq 2$, with vertex at the origin, then the condition

$$\angle \lim_{z\to\tau} \frac{f(z)}{(z-\tau)^{\alpha}} = 0 \tag{3}$$

implies that $f \equiv 0$.

Although the classes $\mathrm{Hol}(\Delta)$ of holomorphic self-mappings of Δ and $\mathrm{Hol}(\Delta, H)$ of functions with positive real part are connected by the composition with the Cayley transform, Theorem A is not a direct consequence of Theorem C, and conversely.

In this note we find rigidity principles for some classes of holomorphic functions produced by continuous dynamical systems, which are related to both $\mathrm{Hol}(\Delta)$ and $\mathrm{Hol}(\Delta, H)$. In particular, by this way one can establish a bridge between Theorems A and C.

We consider, inter alia, the class of mappings $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ which are continuous on $\overline{\Delta}$ and satisfy the boundary flow-invariance condition

$$\mathrm{Re}\, F(z)\overline{z} \leq 1,\ z \in \partial\Delta. \tag{4}$$

In particular, each function $F \in \mathrm{Hol}(\Delta)$ which is continuous on $\overline{\Delta}$ belongs to this class.

Condition (4) can be rewritten in the form

$$\mathrm{Re}\, f(z)\overline{z} \geq 0,\ z \in \partial\Delta, \tag{5}$$

where

$$f(z) = z - F(z). \tag{6}$$

Note that each mapping f satisfying (5) belongs to the class $\mathcal{G}(\Delta)$ of so-called semigroup generators on Δ.

Our main purpose is to establish boundary conditions for a function $f \in \mathcal{G}(\Delta)$ to vanish on Δ identically.

First, we recall that a family $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ is said to be **a one-parameter continuous semigroup on** Δ if

(i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geq 0$,

(ii) $\lim\limits_{t\to 0^+} F_t(z) = z$ for all $z \in \Delta$.

Furthermore, it follows by a result of E. Berkson and H. Porta [5] that each continuous semigroup is differentiable in $t \in \mathbb{R}^+ = [0, \infty)$, (see also [1] and [14]). So, for each continuous semigroup $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$, the limit

$$\lim_{t\to 0^+} \frac{z - F_t(z)}{t} = f(z), \quad z \in \Delta, \tag{7}$$

exists and defines a holomorphic mapping $f \in \mathrm{Hol}(\Delta, \mathbb{C})$. This mapping f is called the **(infinitesimal) generator of** $S = \{F_t\}_{t\geq 0}$. Moreover, the function $u(= u(t,z))$, $(t,z) \in \mathbb{R}^+ \times \Delta$, defined by $u(t,z) = F_t(z)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,z)}{\partial t} + f(u(t,z)) = 0, \\ u(0,z) = z, \quad z \in \Delta. \end{cases} \tag{8}$$

The class of all holomorphic generators on Δ is denoted by $\mathcal{G}(\Delta)$.

Note, that if $F \in \mathrm{Hol}(\Delta)$, then the function $f = I - F$ belongs to $\mathcal{G}(\Delta)$ (see Corollary 3.3.1 in [15]).

The following assertion combines characterizations of the class $\mathcal{G}(\Delta)$ obtained in [2], [3] and [5].

Proposition 1 *Let $f \in \mathrm{Hol}(\Delta, \mathbb{C})$. The following are equivalent:*

(i) f is a semigroup generator on Δ;

(ii) $\mathrm{Re}\, f(z)\overline{z} \geq \mathrm{Re}\, f(0)\overline{z}\,(1 - |z|^2)$ *for all* $z \in \Delta$*;*

(iii) there exists a unique point $\tau \in \overline{\Delta}$ such that

$$f(z) = (z - \tau)(1 - \overline{\tau} z) g(z), \ \ z \in \Delta, \tag{9}$$

where $g \in \mathrm{Hol}(\Delta, \mathbb{C})$, $\mathrm{Re}\, g(z) \geq 0$.

(iv) f admits the representation

$$f(z) = a - \overline{a} z^2 + z p(z),$$

where $a \in \mathbb{C}$ and $p \in \mathrm{Hol}(\Delta, \mathbb{C})$ with $\mathrm{Re}\, p(z) \geq 0$.

Remark 1 *The point τ in (9) is the Denjoy–Wolff point of the semigroup $\{F_t\}_{t\geq 0}$ generated by f. If $\tau \in \Delta$ then $f(0) = 0$ and $\operatorname{Re} f'(\tau) \geq 0$. If $\tau \in \partial\Delta$ then the angular limit $\angle \lim\limits_{z\to\tau} \frac{f(z)}{z-\tau} =: f'(\tau)$ exists and is a nonnegative real number (see [10]).*

2 Rigidity of infinitesimal generators.

Theorem 1 *Let $f \in \mathcal{G}(\Delta)$. Suppose that for some $\tau \in \partial\Delta$*

$$f(z) = a(z-\tau)^3 + o\left(|z-\tau|^3\right)$$

when $z \to \tau$ in each non-tangential approach region at τ. Then $a\tau^2$ is a nonnegative real number. Moreover, $a = 0$ if and only if $f \equiv 0$.

To prove Theorem 1 we need the following lemma.

Lemma 1 *Let $g \in \operatorname{Hol}(\Delta, \overline{H})$. Then for each $\tau \in \partial\Delta$ the limit*

$$k = \angle \lim_{z\to\tau} \frac{g(z)}{1-\overline{\tau}z} \tag{10}$$

is either a nonnegative real number or infinity. Moreover, $g \equiv 0$ if and only if $k = 0$.

Proof. Denote by $C_\tau(z) = \frac{\tau - z}{\tau + z}$ the Cayley transform and set $h = C_\tau^{-1} \circ g \in \operatorname{Hol}(\Delta, \overline{\Delta})$. By the Julia–Wolff–Carathéodory theorem the limit

$$\beta_h = \angle \lim_{z\to\tau} \frac{\tau - h(z)}{\tau - z}$$

exists and is either a nonnegative real number or infinity. Moreover, $\beta_h = 0$ if and only if $h \equiv \tau$.

For any $z \in \Delta$ we have

$$\frac{g(z)}{1-\overline{\tau}z} = \frac{\tau - h(z)}{\tau - z} \cdot \frac{\tau}{\tau + h(z)}\,. \tag{11}$$

Hence, $k = 0$ if and only if $\beta_h = 0$, and therefore $g \equiv 0$.

If β_h is a positive real number, $\beta_h > 0$, then $\angle\lim\limits_{z\to\tau} h(z) = \tau$ and, consequently,

$$k = \angle\lim_{z\to\tau} \frac{\tau - h(z)}{\tau - z} \cdot \angle\lim_{z\to\tau} \frac{\tau}{\tau + h(z)} = \frac{\beta_h}{2} > 0.$$

Let $\beta_h = \infty$. Since $\operatorname{Re}\frac{\tau}{\tau+h(z)} \geq \frac{1}{2}$, formula (11) implies that $k = \infty$. ∎

Alternative proof. If $g \neq 0$, then the function p defined by $p(z) := \frac{1}{g(z)}$ belongs to $\mathrm{Hol}(\Delta, H)$. It is easy to see that for all $\zeta \in \partial\Delta$ the expression $\frac{(1-z\overline{\tau})(1+z\overline{\zeta})}{1-z\overline{\zeta}}$ is bounded on each non-tangential approach region at τ. Then it follows by the Riesz–Herglots formula that

$$\angle\lim_{z\to\tau}(1 - z\overline{\tau})p(z) = \angle\lim_{z\to\tau} \oint\limits_{\partial\Delta} \frac{(1 - z\overline{\tau})(1 + z\overline{\zeta})}{1 - z\overline{\zeta}} dm_p(\zeta) = 2m_p(\tau) \geq 0,$$

where dm_p is a probability measure on $\partial\Delta$. Setting $k = \frac{1}{2m_p(\tau)}$ we get our assertion. ∎

Proof of Theorem 1. Since

$$\angle\lim_{z\to\tau} \frac{f(z)}{z - \tau} = 0,$$

it follows from [10] that $\tau \in \partial\Delta$ is the Denjoy–Wolff point for the semigroup $\{F_t\}_{t\geq 0}$ generated by f. Then by Proposition 1 the function f admits the representation (9):

$$f(z) = (z - \tau)(1 - z\overline{\tau})g(z)$$

with some $g \in \mathrm{Hol}(\Delta, \overline{H})$. Hence, by Lemma 1

$$a\tau^2 = \tau^2 \angle\lim_{z\to\tau} \frac{f(z)}{(z - \tau)^3} = \angle\lim_{z\to\tau} \frac{g(z)}{1 - \overline{\tau}z} = k \geq 0.$$

Obviously, $a = 0$ if and only if $k = 0$. In this case $g \equiv 0$, so $f \equiv 0$. ∎

Corollary 1 (cf. Theorem 5 in [6].) *Let $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ be continuous on $\overline{\Delta}$ and satisfy the boundary condition*

$$\operatorname{Re} F(z)\overline{z} \leq 1,\ z \in \partial\Delta.$$

If F admit the representation

$$F(z) = \tau + (z - \tau) + b(z - \tau)^3 + o\left(|z - \tau|^3\right)$$

when $z \to \tau$ in each non-tangential approach region at some point $\tau \in \partial\Delta$, then $b\tau^2 \leq 0$. Moreover, $b = 0$ if and only if $F \equiv I$.

As a consequence of Lemma 1 we also obtain the following assertion.

Corollary 2 *Let $f \in \mathcal{G}(\Delta)$ be such that $f(\tau) = 0$ for some $\tau \in \partial\Delta$ and $f(0) = a \in \mathbb{C}$. Suppose that f has a finite angular derivative at τ. Then $f'(\tau)$ is a real number with $f'(\tau) \le -2\operatorname{Re}(\overline{a}\tau)$. Moreover, $f'(\tau) = -2\operatorname{Re}(\overline{a}\tau)$ if and only if f generates a group of automorphisms.*

Proof. By Proposition 1 (iv) f admits the representation

$$f(z) = a - \overline{a}z^2 + zp(z), \;\; z \in \Delta, \tag{12}$$

where $p \in \operatorname{Hol}(\Delta, \mathbb{C})$ with $\operatorname{Re} p(z) \ge 0$.

Since $f(\tau) = 0$, we have $p(\tau) = \overline{a}\tau - a\overline{\tau} = 2i\operatorname{Im}(\overline{a}\tau)$ is pure imaginary.

Then it follows from (12), that

$$f'(\tau) = \angle\lim_{z\to\tau} \frac{a - \overline{a}z^2 + zp(z)}{z-\tau} = -2\operatorname{Re}(a\overline{\tau}) + \angle\lim_{z\to\tau} \frac{p(z) - 2i\operatorname{Im}(\overline{a}\tau)}{z\overline{\tau} - 1}\,.$$

Applying Lemma 1 to the function $g(z) = p(z) - 2i\operatorname{Im}(\overline{a}\tau)$, we get $f'(\tau) \le -2\operatorname{Re}(a\overline{\tau})$.

Moreover, $f'(\tau) = -2\operatorname{Re}(a\overline{\tau})$ if and only if $p \equiv 2i\operatorname{Im}(\overline{a}\tau)$, i.e., $f(z) = a + 2i\operatorname{Im}(\overline{a}\tau)\cdot z - \overline{a}z^2$.

By Proposition 3.5.1 in [15] each function of the form $f(z) = a + ibz - \overline{a}z^2$, with $a \in \mathbb{C}$ and $b \in \mathbb{R}$, generates a group of automorphisms of Δ. The proof is complete. ■

Corollary 3 *Let $F \in \operatorname{Hol}(\Delta)$ be such that $F(\tau) = \tau$ and $F(0) = a$, $a \in \Delta$. Suppose that F has a finite angular derivative at τ. Then $F'(\tau) \ge 1 + 2\operatorname{Re}(\overline{a}\tau)$.*

Proof. By a result in [15, Corollary 3.3.1] the function $f(z) = z - F(z)$, $z \in \Delta$ is a generator of a one-parameter semigroup. By our assumptions we have $f(\tau) = 0$ and $f(0) = -a$. Hence, by Corollary 2 $f'(\tau) \le -2\operatorname{Re}(\overline{a}\tau)$, and $F'(\tau) \ge 1 + 2\operatorname{Re}(\overline{a}\tau)$. ■

Now let us consider a class of functions $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ which are continuous on $\overline{\Delta}$ and satisfy the boundary condition

$$\operatorname{Re} f(z)\overline{z} \ge |f(z)| \cos\frac{\alpha\pi}{2} \quad \text{for all} \quad z \in \partial\Delta, \tag{13}$$

for some $\alpha \in (0, 2]$. As we already mentioned if $\alpha \leq 1$ then condition (13) implies $f \in \mathcal{G}(\Delta)$ (cf. Proposition 1 (ii)). Conversely, if $f \in \mathcal{G}(\Delta)$ is continuous on $\overline{\Delta}$, then (13) holds with $\alpha = 1$. So, this class generalize in a sense the class of holomorphic generators which are continuous on $\overline{\Delta}$.

Theorem 2 *Let $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ be continuous on $\overline{\Delta}$ and satisfy the condition (13). Then the condition*

$$\lim_{\substack{z \to \tau \\ z \in \overline{\Delta}}} \frac{f(z)}{(z-\tau)^{2+\alpha}} = 0 \quad \textit{for some} \quad \tau \in \partial\Delta \tag{14}$$

implies that $f \equiv 0$.

Proof. Denote

$$g(z) = \frac{f(z)}{(z-\tau)(1-\overline{\tau}z)}\,.$$

The continuity of f and (14) imply that this function is continuous (consequently, bounded) on $\overline{\Delta}$.

Now we rewrite (13) in the form:

$$-\operatorname{Re}\left[\overline{\tau}(\tau - z)^2 g(z)\overline{z}\right] \geq |\tau - z|^2 \cdot |g(z)| \cdot \cos\frac{\alpha\pi}{2}, \quad z \in \partial\Delta.$$

Hence,

$$\operatorname{Re} g(z) \geq |g(z)| \cdot \cos\frac{\alpha\pi}{2}, \quad z \in \partial\Delta \setminus \{\tau\}.$$

This inequality also holds at the point τ because of the continuity of g.

It follows from the subordination principle for subharmonic functions (see, for example, [11, p. 396]) that the latter inequality holds for all $z \in \overline{\Delta}$. Geometrically this fact means that g maps Δ into the sector $\overline{A_\alpha}$, where

$$A_\alpha = \left\{ w \in \mathbb{C} : |\arg w| < \frac{\alpha\pi}{2},\ \alpha \in (0, 2] \right\}.$$

Suppose that there exists $z \in \Delta$ such that $w = g(z) \in \partial A_\alpha$. Then by the maximum principle $g \equiv \text{const} = w$ and $f(z) = w\tau(z-\tau)^2$. In this case w must be zero, since otherwise we get contradiction with (14). Hence, either $w = 0$ or $g(\Delta) \subset A_\alpha$.

If $w = 0$ then $f \equiv 0$ and we are done.

Let now $g(\Delta) \subset A_\alpha$. Equality (14) implies that

$$\angle \lim_{z \to \tau} \frac{g(z)}{(z-\tau)^\alpha} = -\tau \angle \lim_{z \to \tau} \frac{f(z)}{(z-\tau)^{2+\alpha}} = 0.$$

Applying Theorem C we get $g \equiv 0$, hence $f \equiv 0$. ■

Corollary 4 *Let $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ be continuous on $\overline{\Delta}$ and satisfy the boundary condition*

$$\operatorname{Re} F(z)\overline{z} \le 1 - |F(z) - z| \cos \frac{\alpha\pi}{2}, \quad z \in \partial\Delta, \tag{15}$$

for some $\alpha \in (0, 2]$. If there exists $\tau \in \partial\Delta$ such that

$$F(z) = \tau + (z - \tau) + o(|z - \tau|^{2+\alpha})$$

when $z \to \tau$, then $F \equiv I$.

3 Commuting semigroups.

Theorem 3 *Let f and g be generators of one-parameter commuting semigroups $\{F_t\}_{t \ge 0}$ and $\{G_t\}_{t \ge 0}$, respectively, and $f(\tau) = 0$ at some point $\tau \in \overline{\Delta}$.*

(i) Let $\tau \in \Delta$. If $f'(\tau) = g'(\tau)$ then $f \equiv g$.

(ii) Let $\tau \in \partial\Delta$. Suppose f and g admit the following representations

$$f(z) = f'(\tau)(z - \tau) + \ldots + \frac{f^{(m)}(\tau)}{m!}(z - \tau)^m + o\left(|z - \tau|^m\right) \tag{16}$$

and

$$g(z) = g(\tau) + g'(\tau)(z - \tau) + \ldots + \frac{g^{(m)}(\tau)}{m!}(z - \tau)^m + o\left(|z - \tau|^m\right) \tag{17}$$

when $z \to \tau$ along some curve lying in Δ and ending at τ. If $f^{(m)}(\tau) = g^{(m)}(\tau) \ne 0$, then $f \equiv g$.

Remark 2 *If $\tau \in \partial\Delta$ is the Denjoy–Wolff point of a semigroup generated by a mapping $h \in \mathcal{G}(\Delta)$, then h admits the expansion*

$$h(z) = h'(\tau)(z - \tau) + o(z - \tau)$$

when $z \to \tau$ in each non-tangential approach region at τ and $h'(\tau) = \angle \lim_{z\to\tau} h'(z)$. Moreover, in this case $h'(\tau)$ is a non-negative real number which is zero if and only if h generates a semigroup of parabolic type (see [10]).

Therefore, if f (or g) in Theorem 3 generates a semigroup of hyperbolic type with the Denjoy–Wolff point $\tau \in \partial\Delta$ then the condition $f'(\tau) = g'(\tau)$ is enough to provide that $f \equiv g$.

Remark 3 *As a matter of fact, if f and g have expansion (16) and (17) when $z \to \tau$ in each non-tangential approach region at $\tau \in \partial\Delta$ up to the third order $m = 3$, such that $f'(\tau) = g'(\tau)$, $f''(\tau) = g''(\tau)$ and $f'''(\tau) = g'''(\tau)$ then $f \equiv g$.*

If, in particular, $f^{(i)}(\tau) = g^{(i)}(\tau) = 0$, $i = 1, 2, 3$, then both f and g are equal zero identically by Theorem 1.

Theorem 3 is a consequence of the following more general assertion.

Define two linear semigroups $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ of composition operators on $\mathrm{Hol}(\Delta, \mathbb{C})$ by

$$A_t(h) = h \circ F_t \quad \text{and} \quad B_t(h) = h \circ G_t, \quad t \geq 0. \tag{18}$$

The operators Γ_f and Γ_g defined by

$$\Gamma_f(h) = h'f \quad \text{and} \quad \Gamma_g(h) = h'g \tag{19}$$

are their generators, respectively.

Theorem 4 *Let f and $g \in \mathrm{Hol}(\Delta, \mathbb{C})$ be generators of one-parameter semigroups $\{F_t\}_{t\geq 0}$ and $\{G_t\}_{t\geq 0}$, respectively. Let A_t and B_t be defined by (18). Then the following are equivalent:*

(i) $F_t \circ G_s = G_s \circ F_t$, $s, t \geq 0$, i.e., the semigroups $\{F_t\}_{t\geq 0}$ and $\{G_t\}_{t\geq 0}$ are commuting;

(ii) $A_t \circ B_s = B_s \circ A_t$, $s, t \geq 0$, i.e., the linear semigroups $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ are commuting;

(iii) $\Gamma_f \circ \Gamma_g = \Gamma_g \circ \Gamma_f$, i.e., the linear semigroup generators Γ_f and Γ_g are commuting;

(iv) the Lie commutator

$$[f, g] = f'g - g'f = 0;$$

(v) $f = \alpha g$ for some $\alpha \in \mathbb{C}$.

Proof. Suppose that $f \not\equiv 0$. First we prove the equivalence of assertions (i) and (v).

Let (i) holds. If $f(\tau) = 0$, $\tau \in \Delta$, then τ is a unique common fixed point for the semigroup $\{F_t\}_{t\geq 0}$ generated by f, i.e., $F_t(\tau) = \tau$ for all $t \geq 0$ (see, for example, [5], [15]).

If F_t and G_s are commuting for all $s, t \geq 0$, then we have

$$G_s(\tau) = G_s\left(F_t(\tau)\right) = F_t\left(G_s(\tau)\right).$$

Hence, it follows by the uniqueness of the fixed point τ that $G_s(\tau) = \tau$ for all $s \geq 0$, and so $g(\tau) = 0$.

Consider the function $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ defined by the differential equation

$$\mu h(z) = h'(z) f(z). \tag{20}$$

It is known that if $\mu = f'(\tau)$ then equation (20) has a unique solution $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ normalized by the condition $h'(\tau) = 1$ (see [15]).

In addition, this function h solves Schroeder's functional equation

$$h\left(F_t(z)\right) = e^{-\mu t} h(z). \tag{21}$$

Now, for any $s, t \geq 0$ we get from (21)

$$h\left(G_s\left(F_t(z)\right)\right) = h\left(F_t\left(G_s(z)\right)\right) = e^{-\mu t} h\left(G_s(z)\right).$$

Denote $h_s = h \circ G_s$. Then we have

$$h_s\left(F_t(z)\right) = e^{-\mu t} h_s(z). \tag{22}$$

Differentiating (22) at $t = 0^+$ we get

$$\mu h_s(z) = h_s'(z) f(z). \tag{23}$$

Comparing (20) and (23) implies $h_s(z) = \lambda(s) h(z)$ for some $\lambda(s) \in \mathbb{C}$, or

$$h\left(G_s(z)\right) = \lambda(s) h(z). \tag{24}$$

Since the left-hand side of the latter equality is differentiable in $s \geq 0$, the scalar function $\lambda(s)$ is differentiable too. Differentiating (24) at $s = 0^+$ we get

$$\lambda'(0) h(z) = -h'(z) g(z). \tag{25}$$

Note that $h(\tau) = 0$ while $h(z) \neq 0$ for all $z \in \Delta, z \neq \tau$. In addition, it can be shown (see [15]) that h is univalent. Hence, $h'(z) \neq 0$ for all $z \in \Delta$.

Finally, we obtain from (20) and (25) that

$$f(z) = \alpha g(z), \quad \text{where} \quad \alpha = -\frac{\mu}{\lambda'(0)}\,.$$

Now, let us suppose that f has no null point in Δ. Then the function $p : \Delta \mapsto \mathbb{C}$ given by

$$p(z) = -\int\limits_0^z \frac{d\varsigma}{f(\varsigma)} \tag{26}$$

is well defined holomorphic function on Δ with $p(0) = 0$.

Recall that the semigroup $\{F_t\}_{t\geq 0}$ generated by f can be defined by the Cauchy problem

$$\begin{cases} \dfrac{dF_t(z)}{dt} + f\left(F_t(z)\right) = 0,\ t \geq 0 \\ F_0(z) = z, \quad z \in \Delta \end{cases} \tag{27}$$

Substituting here $f(z) = -\frac{1}{p'(z)}$ we obtain

$$p'\left(F_t(z)\right) dF_t(z) = dt.$$

Integrating the latter equality on the interval $[0, t]$ we get that p is a solution of Abel's functional equation

$$p\left(F_t(z)\right) = p(z) + t. \tag{28}$$

Now, for any fixed $s \geq 0$ we have

$$p\left(G_s\left(F_t(z)\right)\right) = p\left(F_t\left(G_s(z)\right)\right) = p\left(G_s(z)\right) + t.$$

Once again, setting $p_s = p \circ G_s$, we have

$$p_s\left(F_t(z)\right) = p_s(z) + t. \tag{29}$$

Differentiating (29) at $t = 0^+$ we get

$$p_s'(z) = -\frac{1}{f(z)}\,, \tag{30}$$

and by (26), $p_s(z) = p(z) + \kappa(s)$, $\kappa(s) \in \mathbb{C}$, or

$$p\left(G_s(z)\right) = p(z) + \kappa(s). \tag{31}$$

Differentiating (31) at $s = 0^+$ we obtain the equality

$$p'(z) = -\frac{\kappa'(0)}{g(z)}. \tag{32}$$

Comparing (30) and (32) gives

$$f = \alpha g \quad \text{with} \quad \alpha = \frac{1}{\kappa'(0)}. \tag{33}$$

Now we prove that (v)$\Rightarrow$(i). Let $f = \alpha g$ for some $\alpha \in \mathbb{C}$.

First we assume that g has an interior null-point $\tau \in \Delta$. In this case there is a univalent solution of the differential equation

$$\mu h(z) = h'(z) g(z) \tag{34}$$

with some $\mu \in \mathbb{C}$, $\operatorname{Re} \mu \geq 0$.

Since $f = \alpha g$, we have that h is also a solution of the equation

$$\nu h(z) = h'(z) f(z), \quad \nu = \alpha \mu. \tag{35}$$

In turn, equations (34) and (35) are equivalent to Schroeder's functional equations

$$h\left(G_s(z)\right) = e^{-\mu s} h(z), \quad s \geq 0 \tag{36}$$

and

$$h\left(F_t(z)\right) = e^{-\nu t} h(z), \quad t \geq 0, \quad \nu = \alpha \mu, \tag{37}$$

respectively, where $\{F_t\}_{t \geq 0}$ is the semigroup generated by f.

Consequently,

$$\begin{aligned} F_t\left(G_s(z)\right) &= h^{-1}\left(e^{-\nu t} h\left(G_s(z)\right)\right) = h^{-1}\left(e^{-\nu t} \cdot e^{-\mu s} h(z)\right) \\ &= h^{-1}\left(e^{-\mu s} h\left(F_t(z)\right)\right) = G_s\left(F_t(z)\right) \end{aligned}$$

for all $s, t \geq 0$ and we are done.

Now let us assume that g has a boundary null-point $\tau \in \partial \Delta$ with $g'(\tau) \geq 0$ (see Remark 1 above). In this case for each $c \in \mathbb{C}$, $c \neq 0$, Abel's equations

$$p\left(G_s(z)\right) = p(z) + cs$$

and

$$p\left(F_t(z)\right) = p(z) + c\alpha t$$

have the same solution

$$p(z) = -c\int\limits_0^z \frac{d\varsigma}{g(\varsigma)} = -c\alpha\int\limits_0^z \frac{d\varsigma}{f(\varsigma)},$$

which is univalent on Δ.

Once again we calculate

$$\begin{aligned} F_t\left(G_s(z)\right) = p^{-1}\left(p\left(G_s(z)\right) + c\alpha t\right) &= p^{-1}\left(p(z) + c\alpha t + cs\right) \\ &= p^{-1}\left(p\left(F_t(z)\right) + cs\right) = G_s\left(F_t(z)\right). \end{aligned}$$

The implication (v)$\Rightarrow$(i) is proved.

The equivalence of (i) and (ii) is obvious.

To verify the equivalence of (iii) and (iv) we just calculate:

$$\Gamma_f\left(\Gamma_g(h)\right) = h''gf + h'g'f,$$

$$\Gamma_g\left(\Gamma_f(h)\right) = h''fg + h'f'g.$$

Hence, $\Gamma_f \circ \Gamma_g = \Gamma_g \circ \Gamma_f$ if and only if $f'g - g'f = 0$.

Now, it is clear, that (v) implies (iv).

Finally we prove the implication (iv) $\Rightarrow$ (v). Obviously, (iv) implies that if f has no null points in Δ then g also has no null points in Δ and, hence, (v) follows. If $f(\tau) = 0$ for some $\tau \in \Delta$, then also $g(\tau) = 0$, and by (9) one can write $f(z) = (z-\tau)p(z)$ and $g(z) = (z-\tau)q(z)$, where p and q do not vanish in Δ. Now it follows that

$$[f, g] = (z-\tau)[p, q]$$

Hence, again we have $p = aq$, and hence $f = ag$ for some $a \in \mathbb{C}$, $a \neq 0$. ■

Proof of Theorem 3. First we note, that by Theorem 4

$$f = \alpha g, \quad \alpha \in \mathbb{C}. \tag{38}$$

(i) Let $f'(\tau) = g'(\tau) = 0$. By Proposition 1 f admits representation

$$f(z) = (z-\tau)(1-\overline{\tau}z)p(z), \quad z \in \Delta,$$

where $p \in \mathrm{Hol}(\Delta, \mathbb{C})$, $\mathrm{Re}\, p(z) \geq 0$.

Since $f'(\tau) = (1 - |\tau|^2)p(\tau) = 0$, we have $p(\tau) = 0$ and it follows from the maximum principle that $p \equiv 0$. Hence, $f \equiv 0$ and by (38) also $g \equiv 0$.

Assume now $f'(\tau) = g'(\tau) \neq 0$. Then it follows from (38) that $\alpha = 1$ and so $f \equiv g$.

(ii) In general, by (38) we have $f^{(k)}(\tau) = \alpha g^{(k)}(\tau)$, $0 < k \leq m$. Hence, the condition $f^{(k)}(\tau) = g^{(k)}(\tau) \neq 0$ for some $0 < k \leq m$ implies that $\alpha = 1$ and, consequently, $f \equiv g$. ■

Let $\mathcal{S}_f = \{F_t\}_{t \geq 0}$ be the semigroup generated by $f \in \mathcal{G}(\Delta)$. The set $\mathcal{Z}(\mathcal{S}_f)$ of all semigroups $\mathcal{S} = \{G_t\}_{t \geq 0}$ such that

$$F_t \circ G_s = G_s \circ F_t, \quad t, s \geq 0,$$

is called the **centralizer of $\mathcal{S}_f$**.

It is clear that for each $f \in \mathcal{G}(\Delta)$ the centralizer $\mathcal{Z}(\mathcal{S}_f)$ contains $\mathcal{S}_{\alpha f}$ for all $\alpha \geq 0$.

Therefore we will say that **the centralizer of $\mathcal{S}_f$ is trivial** when the conclusion $\mathcal{S} \in \mathcal{Z}(\mathcal{S}_f)$ implies that $\mathcal{S} = \mathcal{S}_{\alpha f}$ for some $\alpha \geq 0$.

Proposition 1 *Let f be the generator of a semigroup $\mathcal{S}_f = \{F_t\}_{t \geq 0}$, and let $\tau \in \partial\Delta$ be the Denjoy–Wolff point of $\mathcal{S}_f$. Then if one of the following conditions holds then the centralizer $\mathcal{Z}(\mathcal{S}_f)$ is trivial:*

(i) $\mathcal{S}_f$ is a hyperbolic type semigroup ($f'(\tau) > 0$) which is not a group;

(ii) f admits the expansion

$$f(z) = a(z - \tau)^3 + o\left((z - \tau)^3\right) \quad \text{with} \quad a \neq 0$$

when $z \to \tau$ in each non-tangential approach region at τ.

The first statement is based on the following simple lemma.

Lemma 2 *Let f and g be generators of two nontrivial (neither f nor g are identically zero) commuting semigroups $\mathcal{S}_f = \{F_t\}_{t \geq 0}$ and $\mathcal{S}_g = \{G_t\}_{t \geq 0}$, respectively. Then $\mathcal{S}_f$ is of hyperbolic type if and only if $\mathcal{S}_g$ is. In this case $f = \alpha g$ with real α. Moreover, $\alpha < 0$ implies that $\mathcal{S}_f$ and $\mathcal{S}_g$ are both groups of hyperbolic automorphisms having 'opposite' fixed points, i.e., the attractive point for $\mathcal{S}_f$ is the repelling point for $\mathcal{S}_g$ and conversely.*

Proof. Since $\mathcal{S}_f$ and $\mathcal{S}_g$ are commuting then by Theorem 4 there exists $\alpha \in \mathbb{C}$ such that $f = \alpha g$. In our settings α is not zero. If τ is the Denjoy–Wolff point of $\mathcal{S}_f$ then $f(\tau) = 0$ and therefore also $g(\tau) = 0$. Now since $f'(\tau) > 0$ then $g'(\tau) = \frac{1}{\alpha} f'(\tau)$ exists finitely and it must be a real number by Corollary 2. So must be α.

Now let us assume that α is negative. Then $g'(\tau) = \frac{1}{\alpha} f'(\tau) < 0$. Hence the semigroup $\mathcal{S}_g$ generated by g must have the Denjoy–Wolff point $\sigma \in \overline{\Delta}$ different from τ.

It is clear that σ can not be inside Δ since otherwise it must be a common fixed point of both semigroups $\mathcal{S}_f$ and $\mathcal{S}_g$ because of the commuting property.

So, $\sigma \in \partial\Delta$ and $g'(\sigma) \geq 0$ (see [10]), then $f(\sigma) = 0$ and $f'(\sigma) \leq 0$. It follows by a result in [16] that

$$0 < f'(\tau) \leq -f'(\sigma) \tag{39}$$

and the equality is possible if and only if f is the generator of a group of hyperbolic automorphisms. By the same theorem we have the reversed inequality for g

$$0 \leq g'(\sigma) \leq -g'(\tau)$$

that means

$$0 \leq \frac{1}{\alpha} f'(\sigma) \leq -\frac{1}{\alpha} f'(\tau).$$

Comparing this inequality with (39) gives us that $f'(\tau) = -f'(\sigma) > 0$ and $g'(\tau) = -g'(\sigma) < 0$ which means that both f and g generate groups of hyperbolic automorphisms with opposite fixed points. ■

Remark. The last assertion of this lemma follows also by a result of Behan (see [4]). Indeed, let $\alpha < 0$. Then the equality $f(z) = \alpha g(z)$ implies that $g'(\tau)$ exists and is a real negative number. So, the Denjoy–Wolff point τ of the semigroup S_f can not be the Denjoy–Wolff point of the semigroup S_g. Hence by [4] we conclude that S_f and S_g are groups of hyperbolic automorphisms.

Proof of Proposition 1. The statement (i) is a direct consequence of the previous lemma. To prove the second statement we note that by Theorem 1 the number $a\tau^2$ is a non-negative real number. On the other hand, since $\mathcal{S}_f$ and $\mathcal{S}_g$ commute by Theorem 4 there is a number $\alpha \in \mathbb{C}$ such that $f = \alpha g$.

Therefore, since $\alpha \neq 0$ also g admits the expansion

$$g(z) = \frac{a}{\alpha} (z-\tau)^3 + o\left((z-\tau)^3\right)$$

and again by Theorem 1 we have that also $\frac{a}{\alpha}\tau^2 \geq 0$. This implies that α is a nonnegative real number. ■

A natural question which arises in the context of the above theorem is:
• If two elements F_{t_0} and G_{s_0} of semigroups $S_f = \{F_t\}_{t\geq 0}$ and $S_g = \{G_t\}_{t\geq 0}$ commute for some positive t_0 and s_0, do these semigroup S_f and S_g commute in the sense:

$$F_t \circ G_s = G_s \circ F_t$$

for each pair $t, s \geq 0$.

The answer is immediately affirmative due to a more general result of C. C. Cowen ([8]), Corollary) if neither F_{t_0} nor G_{s_0}, respectively, are of parabolic type.

The situation becomes more complicated if F_{t_0}, respectively G_{s_0}, are parabolic.

Example 4.4 in [8] shows that there is a triple of such mappings F, G_1 and G_2 in $\mathrm{Hol}(\Delta)$ for which G_1 and G_2 commute with F, but they do not commute each other.

Nevertheless, under some additional requirements on smoothness at the boundary Denjoy-Wolff point repeating the arguments using in the proof of Theorem 1.2 in [17] one can give an affirmative answer the above question. Namely,
• Let F_{t_0} and G_{s_0} be two commuting elements of semigroups S_f and S_g, respectively, $t_0, s_0 > 0$, and let F_{t_0} is of parabolic type with a Denjoy–Wolff point $\tau \in \partial\Delta$. If both F_{t_0} and G_{s_0} belong to the class $C^2(\tau)$ and $F''_{t_0}(\tau)$ as well as $G''_{t_0}(\tau)$ do not vanish, then $f = ag$ for some $a \in \mathbb{C}$, i.e., the semigroups S_f and S_g commute:

$$F_t \circ G_s = G_s \circ F_t$$

for all $t, s \geq 0$.

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